18F05.

Lecture 9.

Adic spaces II.

I. Warmup.

<u>Lemma</u> (André). Let A be a <u>complete</u> Tate ring of char. p. If A is perfect, then A is uniform.

Huber ring: $\overset{\text{open}}{\exists} A_0 \subseteq A$ with topology I-adic for some f.g. $I \subseteq A_0$.

Tate: Huber + unit in $A^{\circ\circ}$: $A^{\circ\circ} \cap A^{\times} \neq \emptyset$.

Tate $\Rightarrow$ topology on any ring of def A_0 is g-adic.

Uniform: A° is a ring of def or eq. A° is bounded.

<u>proof.</u> Let $(A_0,(t))$ be a ring of definition. Set

$$A_n = A_0^{1/p^n} \subseteq A.$$

Then,

$$\begin{tikzcd} A_0 & A_n \arrow[l, hook'] \arrow[d, phantom, "\wr\wr"] \\ & A_0 \arrow[ul, "\varphi^{on}"'] \end{tikzcd}$$

$A_0 \hookleftarrow A_n$ is φ^{on}. $A_\infty := \operatorname{colim} A_n = A_{perf}$.

<u>Claim</u>. (i) $t^{1/p^n} A^\circ \subseteq A_\infty$ for any $n \geq 0$.

(ii) $t^c A_\infty \subseteq A_0$ for some $c \geq 0$.

Fix $f \in A^\circ$. Then, $t^a f^{\mathbb{N}} \subseteq A_0 \subseteq A_\infty$ for some $a \geq 0$. Thus, $t^a f^{p^n} \in A_0$ and so $t^{a/p^n} f \in A_\infty$ for all $n \geq 0$. Here, $t^{a/p^n} f = t^{(a-1)/p^n} t^{1/p^n} f$. For each n, there is some m s.t. $\frac{a}{p^m} \leq \frac{1}{p^n}$, so $\frac{a}{p^m} f \in A_\infty$ implies $t^{1/p^n} f \in A_\infty$.

The claim implies the lemma. Have to show that for every (t^n) there is some (t^m) s.t. $(t^m)A^\circ \subseteq (t^n)$.
Equivalently, have to show that there is some d s.t. $(t^d) A^\circ \subseteq A_0$.
Or, $A^\circ \subseteq t^{-d} A_0$. We have for any $n \geq 0$,

$$t^{1/p^n} t^c A^\circ \subseteq t^c A_\infty \subseteq A_0$$

$$A^\circ \subseteq t^{-c-\frac{1}{p^n}} A_0 \subseteq t^{-c} A_0.$$

Or, just write $t^{c+1} A^\circ \subseteq A_0$, the $n=0$ case of (i).

Prove that A is a Banach space. Then, apply OMT to $\Phi: A \to A$. Get that $t^m A_1 \subseteq A_0$ for some m. Then,

$$t^{\frac{m}{p^n}} A_{n+1} \subseteq A_n$$

for all n. Or,

$$t^{\sum_{i=0}^{n} \frac{m}{p^i}} A_{n+1} \subseteq A_0.$$

So, $t^c A_\infty \subseteq A_0$ for $c \geq \sum_{i=0}^{\infty} \frac{m}{p^i} = \frac{mp}{p-1}$.

II. Valuations.

Def. Γ a valued group. A valuation is a function

$$A \xrightarrow{|\cdot|} \Gamma \amalg \{0\}$$

s.t. (a) $|a+b| \leq \max(|a|,|b|)$,

(b) $|ab| = |a||b|$,

(c) $|0| = 0$, $|1| = 1$.

Notation. $\mathrm{im}(|\cdot|) \cap \Gamma$ generates a subgroup $\Gamma_{|\cdot|}$.

$\mathrm{supp}(|\cdot|) = |\cdot|^{-1}(\{0\})$.

Ex. $|x| = p^{-v_p(x)}$ for $x \in \mathbb{Q}$. $\Gamma_{|\cdot|} \cong p^{\mathbb{Z}} \subseteq \mathbb{R}_{>0}$.

$\mathrm{supp}(|\cdot|) = \{0\}$.

Def. The topology generated by $|\cdot|$ has as basis of nbds of 0 the sets $U_\alpha = \{x \in A : |x| < \alpha\} \cup \mathrm{supp}(|\cdot|)$.

Rem. (b) implies that $\mathrm{supp}(|\cdot|)$ is a prime ideal and the valuation on A induces one on the domain $A/\mathrm{supp}(|\cdot|)$.

Ex. $\mathfrak{p} \subset A$. Define $|\cdot|(x) = \begin{cases} 0 & x \in \mathfrak{p}, \\ 1 & x \notin \mathfrak{p}. \end{cases}$

this is the <u>trivial</u> valuation, written $|\cdot|_{\mathfrak{p}}$.

Def. Two valuations are equivalent if they generate the same topology on A.

III. The valuation spectrum.

Def. A a ring. $\mathrm{Spv}(A) = \{$ equivalence classes of valuations $\}$ with the topology generated by

$$\mathrm{Spv}(A)_{\frac{f}{s}} = \{ |\cdot| \in \mathrm{Spv}(A) : |f| \leq |s| \neq 0 \}$$

$0 \leq \alpha$ $\forall \alpha \in \Gamma$. So, we can have $|f| = 0$, but not $|s| = 0$.

for pairs $f, s \in A$. The <u>valuation spectrum</u>.

<u>Rem.</u> $\mathrm{Spv}(A_{red}) \xrightarrow{\simeq} \mathrm{Spv}(A)$.

<u>Exs.</u> (1) $\mathrm{Spv}(\mathbb{Q}) = \{ |\cdot|_p, |\cdot|_{(0)} \}$. Ostrowski's Theorem.

$$\mathrm{Spv}(\mathbb{Q})_{\frac{m}{n}} = \{ |\cdot| : |m| \leq |n| \}.$$

Every nonempty open contains $|\cdot|_{(0)}$. $\mathrm{Spv}(\mathbb{Q})_{\frac{26}{15}} = \{ |\cdot|_{(0)}, |\cdot|_p : p \neq 3, 5 \}$.

$\mathrm{Spv}(\mathbb{Q}) \cong \mathrm{Spec}(\mathbb{Z})$.

(2) $\mathrm{Spv}(\mathbb{Z}) \xleftarrow{\simeq} \mathrm{Spv}(\mathbb{Q}) \amalg \coprod_p \mathrm{Spv}(\mathbb{F}_p)$.

stratification by support.

$\mathrm{Spv}(\mathbb{F}_p) \cong \{ |\cdot|_{(0)} \} \updownarrow |\cdot|_{(p)}$ in $\mathrm{Spv}(\mathbb{Z})$.

Each $|\cdot|_{(p)}$ is closed.

And, $\overline{\{ |\cdot|_p \}} = \{ |\cdot|_p, |\cdot|_{(p)} \}$.

Lemma. $\mathrm{supp} : \mathrm{Spr}(A) \to \mathrm{Spec}(A)$ is continuous.

proof. Let $f \in A$ and consider the open $\mathrm{Spec}(A_f) \subseteq \mathrm{Spec}\, A$.

Then,
$$\mathrm{supp}^{-1}(\mathrm{Spec}\, A_f) = \{ |\cdot| \in \mathrm{Spr}(A) : f \notin \mathrm{supp}(|\cdot|) \}$$
$$= \{ |\cdot| \in \mathrm{Spr}(A) : |f| \neq 0 \}$$
$$= \mathrm{Spr}(A)_{\frac{0}{f}}.$$

Rem. $\mathrm{supp}^{-1}(\mathfrak{p}) = \mathrm{Spr}(A/\mathfrak{p})$.

Lemma. If $A \xrightarrow{\sigma} B$ is a ring map,
$$\mathrm{Spr}(B) \xrightarrow{\sigma} \mathrm{Spr}(A)$$
is continuous.

proof. Consider $f, s \in A$ and $\mathrm{Spr}(A)_{\frac{f}{s}}$. Then
$$\sigma^{-1}\left(\mathrm{Spr}(A)_{\frac{f}{s}}\right) = \{ |\cdot| \in \mathrm{Spr}(B) : |\sigma(f)| \leq |\sigma(s)| \neq 0 \}$$
$$= \mathrm{Spr}(B)_{\frac{\sigma(f)}{\sigma(s)}}.$$

IV. The adic spectrum.

Note: F-adic in Wedhorn = Huber.

Def. (i) Let A be a Huber ring. A subring $A^+ \subseteq A^\circ$ which is integrally closed is called a ring of integral elts.

(ii) An affinoid ring is a pair (A, A^+) where A is a Huber ring and $A^+ \subseteq A^\circ$ is a ring of int. elts.

Def. If A is a topological ring,

$$\mathrm{Cont}(A) \subseteq \mathrm{Spv}(A)$$

is the subspace of continuous valuations.

$U \subseteq \Gamma \sqcup \{0\}$ is open if either $0 \notin U$ or U contains $\Gamma_{<\gamma}$ for some $\gamma \in \Gamma$.

The subset $\{a \in A : |x| < \gamma\}$ is open for all $\gamma \in \Gamma$.

Def. $\mathrm{Spa}(A, A^+) = \{ |\cdot| \in \mathrm{Cont}(A) :$ [redacted] $|x| \leq 1$ for all $x \in A^+ \}$.